No-go theorem for gravivector and graviscalar on the brane

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Abstract

We prove the no-go theorem that the gravivector ($h_{5\mu}$) and graviscalar (h_{55}) cannot have any on-shell propagation on the Randall-Sundrum (RS) brane. For this purpose, we analyze all of their linearized equations with the de Donder gauge (5D transverse-tracefree gauge). But we do not introduce any matter source. We use the Z_2-symmetry argument and their ($h_{5\mu}, h_{55}$) compatibility conditions with the tensor $h_{\mu\nu}$-equation. It turns out that h_{55} does not have any bulk (massive) and brane (massless) propagations. Although $h_{5\mu}$ has a sort of massive propagations, they do not belong to the physical solution. Hence we confirm that the Randall-Sundrum gauge suffices the on-shell brane physics.

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I. INTRODUCTION

Recently there has been much interest in the Randall-Sundrum brane world [1,2]. The key idea of this model is that our universe may be a brane embedded in higher dimensional space. A concrete model is a single 3-brane embedded in five-dimensional anti-de Sitter space (AdS_5). Randall and Sundrum have shown that the longitudinal part ($h_{\mu\nu}$) of the metric fluctuation satisfies the Schrödinger-like equation with an attractive delta-function [2]. As a result, the massless zero mode which describes the localized gravity on the brane was found.

However, we wish to point out that this has been done with the RS gauge (a 4D transverse-tracefree gauge with $h_{5\mu} = h_{55} = 0$). It seems that this choice may be so restrictive because under this gauge the RS model can describe the tensor fluctuation only. In the Kaluza-Klein approach h_{55} is a 4D scalar and $h_{5\mu}$ is a 4D vector. Hence it is natural to set these fields to be non-zero at the beginning. In order to study the well-defined theory on the brane, it would be better to include the non-zero transverse parts of $h_{5\mu}, h_{55}$. Ivanov and Volovich [3] found that the equation for h_{55} includes a repulsive delta-function. Also the equation of $h_{5\mu}$ has an attractive delta-function potential [4] [1].

In this letter, we analyze all the linearized equations including $h_{5\mu}, h_{55}$ without introducing any matter source. Hence we study the on-shell propagation of their metric fluctuations. We choose initially the de Donder gauge instead of the RS gauge. Consequently we will prove the no-go theorem which states that $h_{5\mu}$ and h_{55} do not have any propagation on the brane.

We start from the second RS model with a positive 3-brane at $z = 0$ [2] in AdS_5 [2–4]

$$\hat{R}_{MN} - \frac{1}{2}\hat{g}_{MN}\hat{R} = \Lambda\hat{g}_{MN} + \sigma\frac{\sqrt{-\hat{g}}}{\sqrt{-\hat{g}_B}}\hat{g}_{B\mu\nu}\delta_M^\mu\delta_N^\nu, \tag{1}$$

which is derived from the action

$$I = \frac{1}{2}\int d^5x\sqrt{-\hat{g}}(\hat{R} + 2\Lambda) + \tilde{\sigma}\int d^4x\sqrt{-\hat{g}_B} \tag{2}$$

with $\kappa_5^2 = 8\pi G_5 = 1$. The RS solution is given in terms of the conformal coordinates (x, z) as

$$ds_{RS}^2 = \hat{g}_{MN}dx^M dx^N = H^{-2}g_{MN}dx^M dx^N \tag{3}$$

with $H = k|z| + 1$ and $g_{MN} = \eta_{MN} = \text{diag}(+ - - - -)$. For $\eta_{MN} = \text{diag}(- + + + +)$ convention, see ref. [9]. In this solution the bulk cosmological constant $\Lambda = -6k^2$ is fine-tuned by the brane tension $\tilde{\sigma} = 6k$. Here the capital indices $M, N, \cdots$ are split into $\mu, \nu, \cdots$ (four-dimensions: $x^\mu = x$) and $5(x^5 = z)$.

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[1]But this can be gauged away. See ref. [5].

[2]If the localized matter of $T_{\mu\nu}(x, z) = \delta(z)T_{\mu\nu}(x)$ is introduced on the brane, the brane is shifted from $z = 0$ to $z = \ell$ [6].

For convenience, it is important to use the conformal transformation of $\hat{g}_{MN} = \Omega^2 g_{MN}$ with $\Omega = H^{-1}$. Introducing a perturbation for the canonical metric $g_{MN} = \eta_{MN} + h_{MN}$, then the line element takes the form

$$ds^2_{RSp} = H^{-2}\left\{(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + 2h_{5\mu}dzdx^\mu + (-1 + h_{55})dz^2\right\}. \tag{4}$$

Before we proceed, it is necessary to comment on the Z_2-symmetry argument. We will use this as a guideline for our analysis. As is easily checked in Eq. (3), this is based on the fact that RS background is symmetric under $z \to -z$. We require that this orbifold symmetry be preserved up to the linearized level for the consistency. In order that ds^2_{RSp} have this symmetry, it requires that $h_{\mu\nu}(x, z), h_{55}(x, z)$ be even with respect to z, but $h_{5\mu}$ be odd : $h_{5\mu}(x, -z) = -h_{5\mu}(x, z)$. This implies that $h_{5\mu}(x, 0) = 0$ on the brane. In other words, we can gauge it away for the brane physics. And h_{55} does not vanish in general. However, this is not the whole story. Actually the propagations around the given background must be determined by solving their linearized equations thoroughly. These equations can be obtained either from a linearized process of the equation (1) or a variational process to the bilinear action which can be obtained from Eq.(2). Here we choose the former one because the latter seems to be a difficult process.

The linearized equation for Eq.(1) takes the form [4]

$$\Box h_{MN} + 3\frac{\partial_K H}{H}\eta^{KL}\left(\partial_N h_{KM} + \partial_M h_{KN} - \partial_K h_{MN}\right)$$
$$-\left(\frac{2\Lambda}{H^2} + \frac{2\sigma}{H}\right)h_{55}\eta_{MN} - \frac{2\sigma}{H}\left\{h_{MN} - \left(h_{\mu\nu} + \frac{h_{55}}{2}\eta_{\mu\nu}\right)\delta^\mu_M\delta^\nu_N\right\} = 0. \tag{5}$$

We stress again that the above equation is suitable for describing the vacuum propagation of metric fluctuations because we do not introduce any localized matter source on the brane and matter source in the bulk. For a localized matter and the RS gauge, see ref. [7,8]. For a bulk matter source, their propagation was discussed in ref. [9]. For our purpose, we choose the de Donder gauge

$$\partial^M h_{MN} = 0, \qquad h^P_P = 0, \tag{6}$$

which means that

$$h^\mu_\mu = h_{55}, \qquad \partial^\mu h_{\mu5} = \partial_5 h_{55}, \qquad \partial^\mu h_{\mu\nu} = \partial_5 h_{5\nu}. \tag{7}$$

The 5D harmonic gauge with $h^P_P \neq 0$ may be useful for the study of the brane world with the bulk matter source [9]. But this gauge is not suitable here. This is because under this gauge, it is not easy to diagonalize the linearized equations to obtain the eigenmodes. From Eq.(5) we obtain three equations

$$\left(\Box - \frac{12k^2}{H^2} - 3f\partial_z\right)h_{55} = 0, \tag{8}$$

$$\left(\Box - \frac{12k}{H}\delta(z)\right)h_{5\mu} - 3f\partial_\mu h_{55} = 0, \tag{9}$$

$$\left(\Box + 3f\partial_z\right)h_{\mu\nu} - 3f\left(\partial_\mu h_{5\nu} + \partial_\nu h_{5\mu}\right) + \frac{12k}{H}\left(\frac{k}{H} - \frac{\delta(z)}{2}\right)h_{55}\eta_{\mu\nu} = 0 \tag{10}$$

with $f = \partial_z H/H = H'/H$.

3

II. GRAVISCALAR PROPAGATION

First let us study the propagation of h_{55}. In analyzing the perturbation, if one encounters a decoupled equation like Eq.(8), one should solve it first. One usually transforms it into the Schrödinger-like equation to get an intuitive understanding. And then one has to check its consistency with the remaining equations (9) and (10). Let us assume $h_{55}(x, z) = H^p(z)\psi_5(z)\hat{h}_{55}(x)$ [3]. For $p = -3/2$, Eq. (8) takes the form

$$\left(\Box_4 + m_5^2\right) \hat{h}_{55}(x) = 0, \tag{11}$$

$$-\frac{1}{2}\psi_5''(z) + V_5(z)\psi_5(z) = \frac{1}{2}m_5^2\psi_5(z) \tag{12}$$

with its potential

$$V_5(z) = \frac{3}{2}\frac{k}{H}\delta(z) - \frac{45}{8}\frac{k^2}{H^2}. \tag{13}$$

We observe that an repulsive delta-function implies the non-existence of localized zero-mode on the brane. To check this thoroughly, we consider $\psi_5^0 = c_5 H^q(z)$ when $m_5^2 = 0$. With $q = 3/2$, Eq. (12) leads to $\psi_5^0 = 0$. Hence we do not find any zero mode for a type of $h_{55}^0 = c_5\hat{h}_{55}(x)$ with constant c_5. This result may be related to those for the zero-mode approach [10–12]. Another possibility is that Eq. (12) may give us a massive propagating solution along z-axis. But considering its compatibility with tensor equation, the answer is No. The tensor equation (10) plays a key role in our analysis because it contains all of metric perturbations. Taking the trace of Eq. (10) and using the gauge condition of Eq. (7) leads to the other equation for h_{55}. Comparing it with Eq. (8), one finds that $h_{55} = 0$ [4]. As a result, we prove the no-go theorem for the graviscalar[3].

III. GRAVIVECTOR PROPAGATION

Now we are in a position to discuss the gravivector propagation. According to the no-go theorem for the gravivector, one does not expect to find any vector propagation on the brane [1,14]. This is based on the Z_2-symmetry argument. Now let us prove this using the linearized equations. Considering $h_{55} = 0$, Eq. (9) becomes a decoupled vector equation. Also we have to solve this equation first too. In order to solve Eq. (9), we introduce the separation of variables as

$$h_{5\mu}(x, z) = \psi_v(z)\hat{h}_{5\mu}(x). \tag{14}$$

Then Eq. (9) leads to

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[3]If there is a delta function source on the brane, $h_{55} = 0$ may break the five-dimensional diffeomorphism and hence the coordinate z becomes discontinuous across the brane. In this case, it is reasonable to start with $h_{55} \neq 0$ [13].

$$\left(\Box_4 + m_v^2\right)\hat{h}_{5\mu}(x) = 0, \tag{15}$$

$$-\frac{1}{2}\psi_v''(z) + V_v(z)\psi_v(z) = \frac{1}{2}m_v^2\psi_v(z) \tag{16}$$

with the attractive $\delta(z)$-type potential $V_v(z)$ as

$$V_v(z) = -\frac{6k}{H}\delta(z). \tag{17}$$

Now let us solve Eq.(16) first. This is just the case of ref. [15]. And then consider its compatibility with the tensor equation. Naively we have two kinds of solutions : one bound state and scattering states.

A. Bound state solution with $m_v^2 < 0$

This solution must satisfy the equation $\psi_v''(z) + m_v^2\psi_v(z) = 0$ with $m_v^2 = -\kappa^2 < 0$ at everywhere, except $z = 0$. Its normalizable solution is

$$\psi_v^{z<0}(z) = Ce^{\kappa z}, \qquad \psi_v^{z>0}(z) = De^{-\kappa z}. \tag{18}$$

It is expected that this solution has the Z_2-symmetry of the RS solution under $z \to -z$ [2]. This can be easily achieved here if $C = D$. Immediately one finds $C = D$ because of the continuity of the wave function at $z = 0$. The derivative of $\psi_v(z)$ is no longer continuous due to the presence of the delta-function. That is, one has

$$\left.\frac{\partial\psi_v}{\partial z}\right|_{z=0^+} - \left.\frac{\partial\psi_v}{\partial z}\right|_{z=0^-} = -12kC, \tag{19}$$

which leads to

$$\kappa = 6k \tag{20}$$

This admits only one of the normalizable bound state solution, irrespective of whatever k is, as

$$\psi_v^e(z) = \sqrt{6k}e^{-6k|z|}. \tag{21}$$

This is an even function with respect to z. However, we do not accept this as a physical solution because it gives us the tachyonic mass of $m_v^2 = -36k^2$ in view of Eq. (15). Furthermore this solution does not satisfy the Z_2-symmetry argument which dictates that $h_{5\mu}$ is odd. Hence we exclude it.

B. Scattering state solution with $m_v^2 > 0$

In this case we start with a plane wave solution

$$\psi_v^{z<0}(z) = Ae^{im_v z} + Be^{-im_v z}, \qquad \psi_v^{z>0}(z) = Fe^{im_v z} + Ge^{-im_v z}. \tag{22}$$

5

The continuity of the wave function at $z = 0$ requires $A + B = F + G$, while the discontinuity of its derivative at $z = 0$ gives $im_v(F - G - A + B) = -12k(A + B)$. If this relation makes sense, one requests that $A + B \neq 0$. Note that we have five unknown quantities of A, B, F, G, m_v but two equations in hand. In this case, at most, we have $F = A + i\beta(A + B)$, $G = B - i\beta(A + B)$ with $\beta = 6k/m_v$. Especially, considering the incident wave propagation($\rightarrow$), we have $G = 0$. This case leads to a conventional scattering with the reflection coefficient $\mathcal{R} = \beta^2/(1 + \beta^2)$ and the transmission coefficient $\mathcal{T} = 1/(1 + \beta^2)$.

However it remains to check whether the above solution is or not consistent with the tensor equation (10). Acting ∂^μ on Eq. (10) and using Eqs. (7) and (9), one gets the compatibility condition with the tensor equation [4,16]

$$(\delta(z)h_{5\mu})' - 3\mathrm{sgn}(z)\delta(z)h_{5\mu} = 0 \tag{23}$$

with the step function of $\mathrm{sgn}(z) = \theta(z)$. This is satisfied trivially at everywhere, except $z = 0$. But $\mathrm{sgn}(z)\delta(z)$ is not well defined at $z = 0$ and thus it requires

$$h_{5\mu}(x, 0) = 0 \rightarrow \quad \psi_v(0) = 0. \tag{24}$$

This is exactly the same requirement from the Z_2-symmetry argument. Here the plane wave solution in Eq. (22) does not satisfy the above compatibility condition because of $A + B \neq 0$. Therefore it cannot to be a solution.

C. An alternative odd solution with $\psi_v(0) = 0$

Finally, we propose an alternative solution which may satisfy Eqs. (16) and (24) simultaneously. This is

$$\psi_v^{z<0}(z) = 2iA \sin m_v z, \quad \psi_v^{z>0}(z) = 2iF \sin m_v z. \tag{25}$$

This can be obtained by demanding an additional condition $\psi_v(0) = 0$ of Eq. (22) : $A + B = F + G = 0 \rightarrow A = -B, F = -G$. Further we have $A = F$ from the discontinuity relation for the derivative of wave function. Then this leads to the propagation of a free particle,

$$\psi_v^o(z) = 2iA \sin(m_v z). \tag{26}$$

We have a few comments on this solution in order. First we get $\mathcal{T} = 1, \mathcal{R} = 0$, which means that it is not the scattering state. Instead it satisfies the boundary condition of $\psi_v^o(0) = 0$ required by both the compatibility and the Z_2-symmetry argument. Also this is an odd function, as required by the symmetry argument. However, importantly, we remind the reader that our background is AdS_5 with the brane. This means that the solution to the linearized equations can carry at least the parameter " k" because the size of AdS_5-box is $1/k$ approximately and the brane tension is $\tilde{\sigma} = 6k$. An example for this is the massive solution for $h_{\mu\nu}(x, z)$ [2]. Unfortunately this wave solution misses "k". It seems that this corresponds to a free particle propagating along z-axis. Due to the compatibility condition (24), this solution $\psi_v^o(z)$ does not account for the presence of the potential at $z = 0$ ($-6k\delta(z)\psi_v(z)$-term in Eq. (16)) appropriately. On the other hand, the bound state solution $\psi_v^e(z)$ in Eq. (21) is obtained by taking into account $\delta(z)\psi_v(z)$ very well. However this does not

satisfy the compatibility condition of $\psi_v(0) = 0$. Also it generates the tachyonic mass for the gravivector. A solution of Eq. (26) is neither a scattering state nor a bound state. Therefore it cannot be regarded as a truly massive propagating solution which accounts for the RS background well. It is obvious that there is no consistent massive solution which satisfies both Eqs. (9) and (10). Lastly, the massless vector propagation is not allowed inherently because with $m_v = 0$, all solutions (21), (22), and (25) imply that there is no wave along z-axis. Even if $m_v \neq 0$, we cannot find any form of $h^o_{5\mu}(x, 0) (\equiv \psi^o_v(0) \hat{h}_{5\mu}(x)) = c_v \hat{h}_{5\mu}(x)$ because of $\psi^o_v(0) = 0$.

All of these results support the symmetry argument for the non-existence of massless gravivector on the brane [1,14]. $h_{5\mu}$ is considered as the gauge degrees of freedom on the brane [5]. Hence if one encounters $h_{5\mu}$, it can be always gauged away without loss of generality.

IV. GRAVITON PROPAGATION

Finally the tensor equation (10) with $h_{55} = h_{5\mu} = 0$ becomes

$$(\Box + 3f\partial_5)h_{\mu\nu} = 0, \tag{27}$$

which just corresponds to the RS case. Introducing the variables $h_{\mu\nu} = H^{3/2}(z)\psi_h(z)\hat{h}_{\mu\nu}(x)$, one finds

$$(\Box_4 + m_h^2)\hat{h}_{\mu\nu} = 0, \quad -\frac{1}{2}\psi_h'' + V_h(z)\psi_h(z) = \frac{1}{2}m_h^2\psi_h(z) \tag{28}$$

with the potential

$$V_h(z) = \frac{15}{8}\frac{k^2}{H^2} - \frac{3}{2}\frac{k}{H}\delta(z). \tag{29}$$

Using the volcano-type potential $V_h(r)$, it is easily to prove $m_h^2 \geq 0$. This fact implies that these are no normalizable bound states for graviton modes. Also the attractive $\delta(z)$-term guarantees the localized zero mode solution $h^0_{\mu\nu} = c_h \hat{h}_{\mu\nu}(x)$ for gravitons at $z = 0$ [2].

V. DISCUSSIONS

We have proved that the gravivector ($h_{5\mu}$) and graviscalar (h_{55}) cannot have any massless mode on the vacuum RS brane. In addition the graviscalar does not have any massive propagation along z-axis. In the case of gravivector, we have a sort of massive propagations. But these cannot be a candidate of the physical solution because they cannot account for the background situation correctly. Our result is based on the Z_2-symmetry argument and their compatibility conditions with the tensor $h_{\mu\nu}$-equation. The first is a guideline for our analysis, whereas the second plays a role as a criterion for selecting a physical solution. Two require the same condition for the gravivector. On the other hand, for the graviscalar, the first implies only that it must be an even function but the second gives us the other equation for h_{55}. Comparing it with Eq. (8) leads to $h_{55} = 0$. Thus the compatibility condition for h_{55} provides a very strong constraint for the non-propagation of the graviscalar on the brane.

Up to now we do not consider the matter source. We comment on the inclusion of matter. There exists a brane-bending effect with the RS gauge which leads to the other massless scalar by introducing the localized matter source on the brane [7,8]. Here the coordinates are discontinuous because the brane appears "bent" in the presence of a localized matter on the brane. Recently, authors in [13,17] carried out the same problem but with alternative gauges for which the coordinates are continuous across the brane.

In conclusion, it is explicitly proved that there is no on-shell propagations for the graviscalar and gravivector. This means that the RS gauge ($\partial^\mu h_{\mu\nu} = h = h_{5\mu} = h_{55} = 0$) suffices the on-shell brane physics when any matter is absent[4]. Even if one introduces a localized matter on the brane, it is reasonable to choose the axial gauge $h_{5\mu} = 0$. This is so because it belongs to the gauge degrees of freedom. But the gauge choice of $h_{55} = 0$ is not still justified when the localized matter is present [13,6].

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[4] But Tanaka [18] insisted that the RS gauge is general even if the matter is absent. Hence, starting with the de Donder gauge, there are still remaining gauge degrees of freedom which eliminate all scalar and vector modes. This does not contradict with our result. In our analysis, if one consider the on-shell degrees of freedom as the physical ones, our mode of Eq.(26) corresponds to a gauge degree of freedom (an off-shell mode).

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